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A PROBLEM OF EVASION FROM MANY PURSUERS

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We show that a controlled point whose velocity is bounded in magnitude can, by remaining in a neighborhood of a given motion, avoid an exact contact with any number of pursuing points whose velocities are less than the velocity of the evading point. We construct a control method ensuring evasion from all pursuers by a finite distance and an arbitrarily small deviation of the point's motion from a given straight line.

1. Statement of the problem. We consider the motions of one evading point E and n pursuing points $P_1, \ldots, P_n$ in an m-dimensional space, $m > 1$. The velocities of all the points can change directions arbitrarily and are bounded in magnitude. The velocity of point E does not exceed a constant v, while the velocities of all pursuers $P_1, \ldots, P_n$ do not exceed kv, where k is a constant, $0 < k < 1$. At the initial instant $t = t_0$ point E is in position E_0 not coinciding with any of the points $P_1, \ldots, P_n$. A ray x passing through point E_0 and a number $\varepsilon > 0$ are specified. The motion of point E along ray x with velocity v is termed nominal. We are required to form a control method for point E, by which this point is at a finite distance from all $P_1, \ldots, P_n$ for all $t \geqslant t_0$ while remaining in the ε-neighborhood of the nominal motion. We assume that at each instant t the velocity of point E can be chosen as a function of the position of points $E, P_1, \ldots, P_n$ on the interval $[t_0, t]$, as well as of the constants v, k, ε and of ray x. For the control method found we are asked to estimate the minimal distance δ from point E to the points $P_1, \ldots, P_n$ for $t \geqslant t_0$.

A control method solving the problem posed is constructed below and the quantity δ is estimated for it. Here the trajectory of point E consists of a finite number of arcs of smooth curves (logarithmic spirals and segments of ray x) and coincides with ray x for fairly large t. To realize the motion it is sufficient to know the positions of points $P_1, \ldots, P_n$ only at those instants that they approach point E by specified distances.

Without loss of generality we can assume $m = 2$, i.e. motion takes place in a plane. Actually, for $m > 2$ we select an arbitrary plane passing through ray x and we consider that point E moves in this plane, evading the projections of points $P_1, \ldots, P_n$ onto this plane. Obviously, the velocities of the projections do not exceed kv. If this evasion problem is solved, then by the same token so is the original problem for $m > 2$. Therefore, below we assume $m = 2$.

2. Evasion from one point. Let us construct the evasion maneuver of point E from one pursuing point P, for which the inequality $EP \geqslant L$ is satisfied for all $t \geqslant t_0$. Here $L > 0$ is an arbitrary given number not exceeding the distance EP at instant t_0. We specify the motion of point E as consisting of three sections; the first and second sections may be of zero length. The velocity of point E on all the sections is constant and equals v. On the first section $[t_0, t_A]$ point E moves along ray x

from the initial point E_0 to a point A at which the equality $EP = L$ first is satisfied. On the second section $[t_A, t_B]$ point E moves along an arc of a certain curve AB whose endpoints lie on ray x, while on the third section $[t_B, \infty]$ it again moves along ray x from point B to ∞.

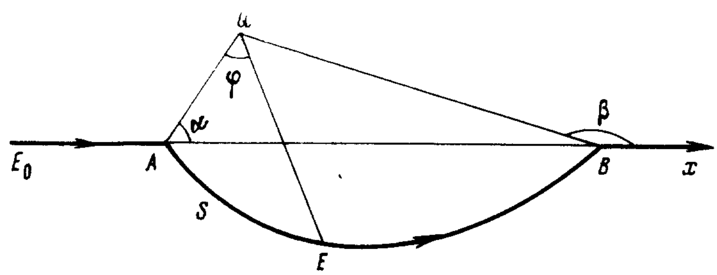

Fig. 1

We find the curve AB from the condition that the inequality $EP \geqslant L$ is fulfilled even if the position of point P for $t > t_A$ is not measured. Let us introduce the following notation: Q is the position of point P at instant t_A; α is the angle between ray x and segment AQ, where $0 \leqslant \alpha \leqslant \pi$; R is the current distance QE ; φ is the current angle between the segments QE and QA, and s is the arc length of curve AE, counted from point A (Fig. 1). Since the velocity of point P does not exceed kv, we have

$$EP \geqslant QE - kv(t - t_A) = R - ks \tag{2.1}$$

The condition $EP \geqslant L$ is satisfied on arc AB if we set

$$R - ks = L \tag{2.2}$$

in (2.1). Differentiating equality (2.2), we obtain

$$dR = kds = k(dR^2 + R^2 d\varphi^2)^{1/2} \tag{2.3}$$

Integrating the differential equation (2,3) under the initial condition $R(0) = L$, we find the function

$$R(\varphi) = Le^{\lambda\varphi} \tag{2.4}$$

Here and later we use the notation

$$\lambda = k(1 - k^2)^{-1/2} = \operatorname{ctg}\gamma, \quad k = \lambda(1 + \lambda^2)^{-1/2} = \cos\gamma \tag{2.5}$$
$$0 < \gamma < \pi/2$$

Using relations (2.4) and (2.5) it is easy to prove that the tangent at point A to the logarithmic spiral (2.4) makes an angle of $\pi - \gamma$ with segment $0A$. Therefore, two cases offer themselves, depending upon the values of $\alpha \in [0, \pi]$. If $\pi - \gamma \leqslant \alpha \leqslant \pi$, then for $\varphi > 0$ the spiral (2.4) in the neighborhood of point A lies to the same side of ray x as does point Q. In this case we set $B = A$; the arc AB shrinks to a point. Here the second section of the motion is absent and point E moves along ray x for the whole time; the angle $\beta = QBx$ equals α and lies within the limits $[\pi - \gamma, \pi]$. When $0 \leqslant \alpha < \pi - \gamma$ the spiral AB intersects ray x at point B (Fig. 1). The angle φ at point B equals $\beta - \alpha$, where $\beta = \angle QBx$. Substituting $\varphi = \beta - \alpha$ and (2.4) into the equation of ray x in the form $R(\varphi)\sin(\varphi + \alpha) =$

$L \sin \alpha$, we obtain a transcendental equation for β

$$f(\beta) = f(\alpha), \quad f(\beta) = e^{\lambda\beta} \sin \beta, \quad \alpha < \beta \leqslant \pi \tag{2.6}$$

The function $f(\beta)$ of (2.6) vanishes at the endpoints of the interval $[0, \pi]$ and, as an investigation shows, increases monotonically on the interval $[0, \pi - \gamma]$ and decreases monotonically on the interval $[\pi - \gamma, \pi]$. Hence it follows that when $0 \leqslant \alpha \leqslant \pi - \gamma$, Eq. (2.6) has in the interval $[\alpha, \pi]$ the unique solution $\beta > \alpha$ lying within the limits $\pi - \gamma < \beta \leqslant \pi$. Thus, in both cases, i.e. for any $\alpha \in [0, \pi]$, we have, with due regard to notation (2.5),

$$\pi - \gamma \leqslant \beta \leqslant \pi, \quad \cos \beta \leqslant -k \tag{2.7}$$

The inequality $EP \geqslant L$ is satisfied by construction on the first two sections ($t_0 \leqslant t \leqslant t_B$) for the evasion maneuver described. For an arbitrary instant $t > t_B$ of the third section we have

$$EP \geqslant QE - kv(t - t_A) = (QB^2 + EB^2 - 2EB \cdot QB \cos \beta)^{1/2} - \tag{2.8}$$
$$kv(t - t_B) - kv(t_B - t_A) \geqslant QB - EB \cos \beta - kEB - kS_{AB}$$

According to (2.2) the length S_{AB} of arc AB equals $k^{-1}(QB - L)$. Using now inequality (2.7), from (2.8) we have that $EP \geqslant L$. Therefore, in all cases the maneuver constructed ensures the inequality $EP \geqslant L$ for all $t \geqslant t_0$.

3. Maneuver for evasion from n points. We construct the proposed method of evading n pursuers on the basis of the maneuver of Sect. 2. By δ_0 we denote the minimal one of two distances $EP_1, \ldots, EP_n$ at the instant t_0; by hypothesis, $\delta_0 > 0$. The motion of point E depends upon parameters L and $\varkappa$ such that $0 < L \leqslant \delta_0$ and $0 < \varkappa < 1$; these parameters will be chosen below. We introduce the notation $L_j = L\varkappa^{j-1}$ and we call the instant t_j, when the condition

$$\min_i EP_i = L_j \equiv L\varkappa^{j-1}, \quad i = 1, \ldots, n, \; j = 1, 2, \ldots, 0 < \varkappa < 1 \tag{3.1}$$

is first satisfied after the start of motion, the instant of j-th encounter.

We specify the motion of point E in the following manner: at each instant t point E moves at a velocity v, constant in magnitude, along a program trajectory for the given instant t. Let us define the concept of a program trajectory for each instant $t \geqslant t_0$. For any $t \geqslant t_0$ the current program trajectory is an oriented piecewise-smooth curve without selfintersections, starting from the current position of point E at instant t and going off to infinity along arc x. For $t = t_0$ the program trajectory is ray x. On the intervals $t_j < t < t_{j+1}$, $j = 0, 1, \ldots$ the origin of the program trajectory is moving along it together with point E; here the program trajectory is not altered in other respects. At the encounter instants t_j, $j = 1, 2, \ldots$ the program trajectory is reconstructed in the following way. By A_j and Q_j we denote, respectively, the positions at instant t_j of point E and of that one of points P_i for which the minimum in relation (3.1) is achieved. If for $t = t_j$ the minimum in (3.1) is achieved simultaneously for several points P_i, then as Q_j we select, for definiteness, that one of them for which the number i is the least. We draw two L_j-spirals whose equations have the form

$$R_j = L_j \exp(\lambda\varphi_j), \quad 0 \leqslant \varphi_j \leqslant \pi, \quad j = 1, 2, \ldots \tag{3.2}$$

Here R_j is the current distance from Q_j; φ_j is the polar angle read off from the straight line Q_jA_j in two opposite directions for the two spirals being examined. The arcs of the L_j-spirals (3.2) are mirror-symmetric to each other relative to segment Q_jA_j and have common endpoints for $\varphi_j = 0$ and $\varphi_j = \pi$.

Suppose that the program trajectory has been constructed for $t = t_j - 0$ and that it starts at point A_j. If for $t = t_j - 0$ the program trajectory has no other common points besides A_j with the L_j-spirals (3.2) constructed, then the program trajectory for $t = t_j + 0$ will be the same as that for $t = t_j - 0$. Otherwise, by B_j we denote the first point after A_j of the intersection of the program trajectory corresponding to $t_j - 0$ with the closed curve formed by the arcs of the L_j-spirals (3.2). The program trajectory corresponding to $t = t_j + 0$ is the curve comprised of the arc A_jB_j of that one of the L_j-spirals containing point B_j and of the remainder of the program trajectory for $t_j - 0$ starting from point B_j.

The process described recurrently determines the program trajectory for any instant $t \geqslant t_0$ for any finite number of encounters. For $t \in (t_j, t_{j+1})$ the program trajectory consists, by construction, of arcs of $L_j, L_{j-1}, \ldots, L_1$-spirals, connected in the order of decreasing index, and of the portion of ray x which includes the point at infinity. The arcs of all the spirals correspond to the polar angles $0 \leqslant \varphi \leqslant \pi$, but some arcs can be absent. The construction of a program trajectory at any instant completely determines the control method for point E. The real trajectory of point E consists of arcs of L_j-spirals and of a portion of ray x, and to accomplish this trajectory it is sufficient to measure the positions of points P_i only at the encounter instants. It remains to select the parameters L and $\varkappa$ so as to ensure the finiteness of the number of encounters and the evasion of point E from all $P_1, \ldots, P_n$ while its motion remains in the ε-neighborhood of the nominal motion.

4. Estimate of the distances. At first we assume certain estimates. Suppose that immediately before the j-th encounter ($t = t_j - 0$), $j \geqslant 1$ the program trajectory starts off on an arc of an L_p-spiral, $p \leqslant j - 1$, of nonzero length, after which there follows an arc of an L_q-spiral, $q \leqslant p - 1$. Figure 2 shows an arc A_pB_p of an L_p-spiral with pole Q_p and the arcs of two L_j-spirals with pole Q_j. As a result of the construction in Sect. 3 we obtain a program trajectory for the instant $t = t_j + 0$, a section $A_jB_jB_p$ of which is shown in Fig. 2 by a heavy line with arrows. For $t \geqslant t_j$ we estimate the distance EP_i from point E up to that point P_i with which the j-th encounter occurred. At first we assume that the next $(j + 1)$-st encounter does not take place so long as point E moves over the section $A_jB_jB_p$ of the program trajectory. On the arc A_jB_j of the L_j-spiral we have $EP_i \geqslant L_j$ in accord with the property of a logarithmic spiral (see Sect. 2). Let us estimate in two ways the distance EP_i as point E moves along the arc B_jB_p.

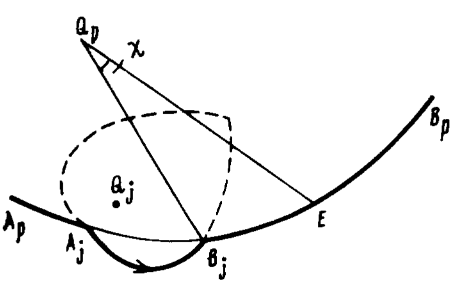

Fig. 2

We introduce into consideration a point E' moving with a constant velocity v along the straight line A_jB_j from the point B_j in the opposite direction of A_j. Here we suppose that the point E' is located at B_j at the very same instant t' as point E. Applying to points E' and P_i the reasonings presented at the end of Sect. 2 for the points E and P, we obtain that $E'P_i \geqslant L_j$ for $t \geqslant t'$. By the triangle inequality we have

$$EP_i \geqslant E'P_i - EE' \geqslant L_j - EE' \tag{4.1}$$

Since points E and E' have a velocity v of like magnitude and coincide at the instant t', we obtain

$$EE' = \left| v \int_{t'}^{t} [\mathbf{e}(t) - \mathbf{e}']\, dt \right|, \quad t \geqslant t' \tag{4.2}$$

Here $\mathbf{e}'$ is the unit vector of the straight line A_jB_j and $\mathbf{e}(t)$ is the unit vector of the tangent to arc B_jB_p. By s we denote the current length of an arc of curve B_jB_p, counted off from point B_j. We have

$$|\mathbf{e}(t) - \mathbf{e}'| = \left| \mathbf{e}(t') - \mathbf{e}' + \int_{t'}^{t} \frac{d\mathbf{e}}{dt} dt \right| \leqslant |\mathbf{e}(t') - \mathbf{e}'| + \int_0^s \left| \frac{de}{ds} \right| ds \tag{4.3}$$

The unit vector $\mathbf{e}'$ of the chord A_jB_j equals the unit vector of the tangent at some point of the L_p-spiral, lying between A_j and B_j. By s' we denote the length of the arc of the L_p-spiral from this intermediate point to point B_j; from (4.3) we obtain

$$|\mathbf{e}(t) - \mathbf{e}'| \leqslant \int_0^{s'} \left| \frac{de}{ds} \right| ds + \int_0^{s} \left| \frac{de}{ds} \right| ds \leqslant \frac{s + s'}{\rho_m} < \frac{s + s''}{\rho_m} \tag{4.4}$$

Here we have used the inequality $|de / ds| \leqslant \rho_m^{-1}$, where ρ_m is the minimal radius of curvature of the L_p-spiral and s'' is the length of the arc A_jB_j of this spiral. The radius of curvature of the L_p-spiral defined by Eq. (3.2) is

$$\rho = (R^2 + R_\varphi^2)^{3/2} (R^2 + 2R_\varphi^2 - RR_{\varphi\varphi})^{-1} = L_p (\lambda^2 + 1)^{1/2} e^{\lambda\varphi}$$

Here the subscripts φ denote differentiation with respect to φ_p, while the subscript p has been dropped.

For $\varphi \geqslant 0$ the minimal radius of curvature is

$$\rho_m = L_p \sqrt{\lambda^2 + 1} \tag{4.5}$$

We estimate the length s'' of the arc A_jB_j of the L_p-spiral using relation (2.2) for the L_p-spiral and the triangle inequality

$$s'' = (Q_pB_j - Q_pA_j)\, k^{-1} \leqslant A_jB_jk^{-1} \leqslant (Q_jB_j + Q_jA_j)k^{-1} \leqslant L_jk^{-1} (e^{\lambda\pi} + 1) \tag{4.6}$$

In the estimate (4.6) we have used the relations

$$Q_jA_j = L_j, \quad Q_jB_j \leqslant L_je^{\lambda\pi} \tag{4.7}$$

following from (3.2). Substituting inequality (4.4) into (4.2) and integrating, we obtain

$$EE' < (1/_2 s^2 + ss'') / \rho_m \tag{4.8}$$

Introducing relations (4.8), (4.5) and (4.6) into inequality (4.1) and using notation (2.5), we find

$$EP_i > L_j - L_p^{-1}a_1s^2 - L_jL_p^{-1}a_2s, \quad s \geqslant 0 \tag{4.9}$$

$$a_1 = 1/_2 (\lambda^2 + 1)^{-1/2}, \quad a_2 = \lambda^{-1} (e^{\lambda\pi} + 1)$$

For an estimate of the distance EP_i in another way we note that point P_i occupies position Q_j at $t = t_j$ and its velocity does not exceed kv. Therefore,

$$EP_i \geqslant Q_jE - Q_jP_i \geqslant (B_jE - Q_jB_j) - kv\,(t - t_j) \geqslant \qquad (4.10)$$
$$B_jE - L_je^{\lambda\pi} - k\,(s_j + s)$$

Here we have twice used the triangle inequality for the estimates of EP_i and Q_jE, as well as the inequality (4.7) for Q_jB_j. The length of the arc A_jB_j of the L_j-spiral is denoted by s_j. We estimate it, taking equality (2.2) and estimates (4.7) into account

$$s_j = k^{-1}\,(Q_jB_j - Q_jA_j) \leqslant L_jk^{-1}\,(e^{\lambda\pi} - 1) \qquad (4.11)$$

Denoting

$$R = Q_pE, \quad R_* = Q_pB_j, \quad \chi = \angle B_jQ_pE$$

for brevity and using the equalities $R = R_*e^{\lambda\chi} = R_* + ks$ stemming from relations (3.2) and (2.2) for the L_p-spiral, we obtain (see Fig. 2)

$$B_jE = (R^2 + R_*^2 - 2RR_*\cos\chi)^{1/2} = (R - R_*)\ [1 + \qquad (4.12)$$
$$4\sin^2(\chi/2)\,RR_*\,(R - R_*)^{-2}]^{1/2} = ks\,[1 + 4\sin^2(\chi/2)\,e^{\lambda\chi}\times$$
$$(e^{\lambda\chi} - 1)^{-2}]^{1/2} = ks\,[1 + \sin^2(\chi/2)\,\mathrm{sh}^{-2}(\lambda\chi/2)]^{1/2} \quad (0 \leqslant \chi \leqslant \pi)$$

It is easy to verify by differentiation that on the interval $0 \leqslant \chi \leqslant \pi$ the function $\sin(\chi/2)\,\mathrm{sh}^{-1}(\lambda\chi/2)$ decreases monotonically for any $\lambda > 0$ and, therefore, achieves its minimum for $\chi = \pi$. Then from (4.12) we obtain

$$B_jE \geqslant ks\,[1 + \mathrm{sh}^{-2}(\lambda\pi/2)]^{1/2} = ks\,\mathrm{cth}\,(\lambda\pi/2) \qquad (4.13)$$

Introducing inequalities (4.11) and (4.13) into inequality (4.10) and using notation (2.5), we find

$$EP_i \geqslant a_3s - a_4L_j, \quad s \geqslant 0 \qquad (4.14)$$
$$a_3 = k\,[\mathrm{cth}\,(\lambda\pi/2) - 1] = 2\lambda\,(1 + \lambda^2)^{-1/2}\,(e^{\lambda\pi} - 1)^{-1}$$
$$a_4 = 2e^{\lambda\pi} - 1 > 1$$

Comparing the two estimates (4.9) and (4.14), we obtain

$$EP_i \geqslant \max\,[f_1(s),\ f_2(s)] \geqslant \min_{s\geqslant 0}\,\max\,[f_1(s),\ f_2(s)] \qquad (4.15)$$
$$f_1(s) = L_j - L_p^{-1}a_1s^2 - L_jL_p^{-1}a_2s, \quad f_2(s) = a_3s - a_4L_j$$

The function $f_1(s)$ decreases monotonically while the function $f_2(s)$ increases monotonically with the growth of s for $s \geqslant 0$; $f_1(0) > 0$ and $f_2(0) < 0$. The minimum in (4.15) is reached for the $s_* > 0$ for which

$$f_1(s_*) = f_2(s_*), \quad EP_i \geqslant f_2(s_*) = L_j\mu \qquad (4.16)$$

Here μ is a dimensionless quantity introduced by the last relation in (4.16). From this relation, using (4.15), we express

$$s_* = L_j\,(a_4 + \mu)\,a_3^{-1} \qquad (4.17)$$

We substitute equality (4.17) and the expressions (4.15) for f_1 and f_2 into the first equation of (4.16) and, next, we solve the resulting equation relative to L_j. We obtain

$$\frac{L_j}{L_p} = g(\mu), \quad g(\mu) = \frac{a_3^2\,(1 - \mu)}{(a_4 + \mu)\,(a_1a_4 + a_2a_3 + a_1\mu)} \qquad (4.18)$$

The function $g(\mu)$ decreases strictly on the interval $[0, 1]$; consequently, the inverse function g^{-1} exists which is continuous and decreases strictly on the interval $[0, g(0)]$. Therefore, allowing further for relations (3.1) and for the inequality $j - p \geqslant 1$, we obtain

$$\mu = g^{-1}(\varkappa^{j-p}) \geqslant g^{-1}(\varkappa), \quad 0 < \varkappa < g(0) \tag{4.19}$$

We note that

$$g(0) < a_3^2 a_4^{-2} a_1^{-1} < a_3^2 a_1^{-1} = 8\lambda^2 (1 + \lambda^2)^{-1/2} (e^{\lambda\pi} - 1)^{-2} <$$
$$8[\lambda / (e^{\lambda\pi} - 1)]^2 < 8\pi^{-2} < 1 \quad (\lambda > 0)$$

follows from relations (4.18), (4.9) and (4.14). Substituting inequality (4.19) into(4.16), we have

$$EP_i \geqslant L_j g^{-1}(\varkappa) \tag{4.20}$$

Thus, as point E moves along the arc B_jB_p the estimate (4.20) is valid for any $\varkappa$ from the interval

$$0 < \varkappa < g(0) < 1 \tag{4.21}$$

Since under conditions (4.21) we have $g^{-1}(\varkappa) < 1$, inequality (4.20) is valid as well for motion along the arc A_jB_j of the L_j-spiral, where $EP_i \geqslant L_j$. The fulfillment of conditions (4.21) guarantees estimate (4.20) as point E moves from A_j up to its departure at the point B_p on the arc of the L_q-spiral, $q \leqslant p - 1$. This assertion is valid, of course, also when one or both arcs A_jB_j and B_jB_p are zero.

We assume above that the next $(j + 1)$-st encounter does not occur as point E moves along the section $A_jB_jB_p$. We now relinquish this assumption. Let the point E experience after the instant t_j encounters with the points $P_1, \ldots, P_n$ and let $\nu(t) \geqslant 0$ be the number of these encounters on the interval (t_j, t). Let τ denote the instant when point E first goes onto the program trajectory corresponding to the instant $t_j + 0$ after point B_p. In other words, τ is the first instant after t_j that point E goes onto some L_r-spiral, $r \leqslant q \leqslant p - 1$. Let us consider a point E_* moving at the velocity v on the section $A_jB_jB_p$ of the program trajectory. Suppose that point E_* coincides with E at instant t_j and arrives at point B_p at instant τ_*. Then, in accord with the triangle inequality and the estimate (4.20) obtained for the point E_*, we have

$$EP_i \geqslant E_*P_i - EE_* \geqslant L_j g^{-1}(\varkappa) - EE_*, \quad t_j \leqslant t \leqslant \tau_* \tag{4.22}$$

Let us estimate the distance between points E and E_*

$$EE_* = \left| \int_{t_j}^{t} [\mathbf{v}(t) - \mathbf{v}_*(t)]\, dt \right|, \quad t_j \leqslant t \leqslant \tau_* \tag{4.23}$$

Here $\mathbf{v}(t)$ and $\mathbf{v}_*(t)$ are the velocity vectors of points E and E_*, respectively, equal in magnitude to a constant v. For the sections of the trajectory of point E, belonging to the trajectory of point E_*, the corresponding contribution to integral (4.23) equals zero. For the remaining sections integral (4.23) is majorized by twice their total length, i.e.,

$$EE_* \leqslant 2\Sigma, \quad t_j \leqslant t \leqslant \tau_* \tag{4.24}$$

Here Σ is the sum of the arcs of the $L_{j+1}, \ldots, L_{j+\nu}$-spirals, which we estimate by means of inequality (4.11) and formula (3.1)

$$\Sigma \leqslant s_{j+1} + s_{j+2} + \ldots + s_{j+\nu} \leqslant L_j k^{-1} (e^{\lambda\pi} - 1) \varkappa (1 - \varkappa^\nu)(1 - \varkappa)^{-1} \tag{4.25}$$

If $\tau \leqslant \tau_*$, estimates (4.22) and (4.24) are valid on the whole interval $[t_j, \tau]$. If, however, $\tau > \tau_*$, a further consideration of $t \in [\tau_*, \tau]$ is needed. The path traversed by point E in the time $[t_j, t]$ does not exceed the path traversed by point E_* in the time $[t_j, \tau_*]$ by more than Σ . Therefore, point E traverses a path no greater than Σ in the time $t - \tau_*$ and $t - \tau_* \leqslant \Sigma / v$. During this time the distance EP_i can decrease by no more than

$$v(1+k)(t-\tau_*) \leqslant (1+k)\Sigma$$

Subtracting this quantity from the right-hand side of inequality (4.22) and using the inequalities (4.24) and (4.25), we obtain the required estimate

$$EP_i \geqslant L_j g^{-1}(\varkappa) - (3+k)\Sigma \geqslant L_j [g^{-1}(\varkappa) - (3k^{-1} + 1)(e^{\lambda\pi} - 1)\varkappa(1-\varkappa^\nu)(1-\varkappa)^{-1}], \quad t_j \leqslant t \leqslant \tau \tag{4.26}$$

Note that the boundedness of Σ follows from formula (4.25) and, consequently, that of τ as $\nu \to \infty$.

5. Choice of the maneuver parameters. Let us require that the inequality

$$g^{-1}(\varkappa) \geqslant (3k^{-1} + 1)(e^{\lambda\pi} - 1)\varkappa(1-\varkappa^\nu)(1-\varkappa)^{-1} + \varkappa^{\nu+1} \tag{5.1}$$

be satisfied for all integers ν . Under condition (5.1), from inequality (4.26) follows $EP_i \geqslant L_{j+\nu+1}$ for all $t_j \leqslant t \leqslant \tau$. This implies that among the encounters taking place in the interval (t_j, t) there is no encounter with point P_i. Therefore, under condition (5.1), encounters with point P_i do not occur on the whole interval (t_j, τ) . We rewrite condition (5.1) as

$$g^{-1}(\varkappa) \geqslant b\varkappa - \varkappa^{\nu+1}(b-1), \quad b = (3k^{-1} + 1)(e^{\lambda\pi} - 1)(1-\varkappa)^{-1} \tag{5.2}$$

Let us estimate quantity b, using formulas (2.5). We obtain

$$b > 3k^{-1}\lambda\pi > 3\pi > 1$$

Therefore, inequality (5.2) is satisfied for all $\nu \geqslant 0$ if it is satisfied for $\nu = 0$. Substituting $\nu = 0$ into (5.2), we obtain the condition $g^{-1}(\varkappa) \geqslant \varkappa$. Combining this with the condition (4.21) found above and allowing for the monotonicity of function g, we have

$$0 < \varkappa \leqslant \varkappa_* \tag{5.3}$$

where $\varkappa_*$ is the single positive root of the equation

$$g(\varkappa_*) = \varkappa_*, \quad \varkappa_* > 0 \tag{5.4}$$

We select the parameter $\varkappa$ from interval (5.3). Here the motion of point E described in Sect. 3 possesses the following property. If the j-th encounter took place with point P_i as point E moved along an arc of the L_p-spiral, $p \leqslant j - 1$, then the next encounter with this same point P_i can take place no earlier than after point E has gone onto an arc of the L_r-spiral, $r \leqslant p - 1$. This property extends to ray x which can be taken as an L_0-spiral.

Without loss of generality suppose that the points P_i have been numbered in the order in which their first encounters with point E takes place. Then, the first encounter with point P_1 occurs on ray x and, in accordance with the property established, there are no other encounters with this point. The first encounter with point P_2 can take place either on an arc of the L_1-spiral or on ray x after leaving this arc. In the first case a repeated encounter with point P_2 can occur only after going onto ray x, while in the second case it does not occur at all. Let us estimate the total number $N(n)$ of encounters with with n pursuers. From the preceding arguments follows $N(1) \leqslant 1$ and $N(2) \leqslant 3$.

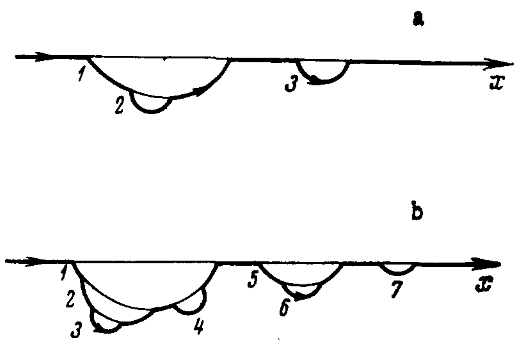

Fig. 3

Figure 3 shows schematically, not to scale, typical trajectories of point E for (a) $n = 2$, $N = 3$ and (b) $n = 3$, $N = 7$; the digits indicate the encounter numbers.

The following inequality, provable by induction, is valid:

$$N(n) \leqslant 2^n - 1 \qquad (5.5)$$

Inequality (5.5) is true for $n = 1, 2, 3$. Let it be true for n. Then, in the case of $n+1$ pursuers the total number of encounters can be estimated as follows. After the first encounter with point P_1 the point E moves along the L_1-spiral and can encounter the remaining n points. Not more than $N(n)$ encounters with these points can occur before going onto ray x . After going onto ray x point P_1 need not be taken into account since encounters with it do not occur any more, but $N(n)$ further encounters can take place with the remaining points. Thus, with due regard to (5.5)

$$N(n+1) \leqslant 1 + N(n) + N(n) \leqslant 2^{n+1} - 1$$

and inequality (5.5) has been proved by induction.

Let us now estimate the distance between point E and the point E° accomplishing the nominal motion. Analogously to inequalities (4.23) — (4.25) we obtain

$$EE^\circ \leqslant 2\,[s_1 + s_2 + \ldots + s_{N(n)}] < 2Lk^{-1}(e^{\lambda\pi} - 1)\;(1 - \varkappa)^{-1} \qquad (5.6)$$

The motion of point E must lie in the ε-neighborhood of the nominal motion, i.e., $EE^\circ \leqslant \varepsilon$. For this, according to (5.6), it suffices to take

$$0 < L \leqslant \min\,[{}^1/_2\,\varepsilon k\,(1 - \varkappa)\;(e^{\lambda\pi} - 1)^{-1},\;\delta_0] \qquad (5.7)$$

We have here also allowed for the condition $L \leqslant \delta_0$ imposed above. Thus the parameters L and $\varkappa$ must be selected within the limits (5.3) and (5.7). Here the evasion maneuver of Sect. 3 satisfies all the conditions imposed and the number of encounters satisfies inequality (5.5).

According to (4.18) and (5.4) the determination of $\varkappa_*$ requires solving a cubic equation. Let us obtain a simple explicit expression for $\varkappa = \varkappa_0$ lying within limits (5.3). Together with the function g from (4.18) we consider the linear function

$$g_0(\mu) = (1 - \mu)\,g_1, \quad g_1 = a_3^2\,(a_4 + 1)^{-1}(a_1 a_4 + a_2 a_3 + 1)^{-1} \qquad (5.8)$$

Comparing (4.18) and (5.8) we see that $g_0(\mu) \leqslant g(\mu)$ for $0 \leqslant \mu \leqslant 1$, and the root $\varkappa_0$ of the equation

$$g_0(\varkappa_0) = \varkappa_0 \qquad (5.9)$$

lies within the limits (5.3). Solving Eq. (5.9) with due regard to (5.8), we find the quantity required

$$\varkappa_0 = g_1 / (1 + g_1) \qquad (5.10)$$

Let us estimate further the minimal distance δ between the points E and $P_1, \ldots, P_n$ for $t > t_0$. Since the maximum number of encounters does not exceed $N(n)$

from (5.5)

$$\delta = \min_{i,t} EP_i \geqslant L_{N(n)+1} = L\varkappa^{N(n)} \geqslant L\varkappa^{(2^n-1)} \quad (5.11)$$
$$1 \leqslant i \leqslant n, \quad t \geqslant t_0$$

If we select the maximum possible L allowed by inequality (5.7), then from (5.11) we obtain

$$\delta \geqslant \min [C_n(k)\varepsilon, \; C_n^*(k)\delta_0] \quad (5.12)$$
$$C_n(k) = {}^1/_2\, k(1-\varkappa)(e^{\lambda\pi}-1)^{-1}\varkappa^{(2^n-1)}, \qquad C_n^*(k) = \varkappa^{(2^n-1)}$$

As $\varkappa$ we can take any number from interval (5.3), for example, the $\varkappa_0$ from (5.10). We obtain explicit expressions for $\varkappa_0$, $C_n(k)$ and $C_n^*(k)$ by substituting relations (5.8), (4.9), (4.14) and (2.5) into formulas (5.10) and (5.12). In particular, when the capabilities of the pursuers approach the capability of the evading point $(k \to 1)$, we find according to the formulas indicated above

$$\varkappa_0 \approx 2\lambda e^{-4\pi\lambda}, \quad C_n(k) \approx 0.5 e^{-\pi\lambda}\varkappa_0^{(2^n-1)}$$
$$C_n^*(k) \approx \varkappa_0^{(2^n-1)}, \quad \lambda = k(1-k^2)^{-1/2} \to \infty, \quad k \to 1$$

We note that the evasion strategy proposed in Sect. 3 for point E, as well as the bounds (5.3) and (5.7) on the choice of parameters L and $\varkappa$, do not depend upon the number n of pursuers.

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ON A PROBLEM OF l-ESCAPE

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We give criteria for escape and l-escape in nonlinear differential games. The paper is closely related to the investigations in [1 — 8].

1. Let the motion of a vector z in an n-dimensional Euclidean space R be described by the vector differential equation

$$dz/dt = f(t, z, u, v) \quad (1.1)$$

where $t \geqslant 0$; $u \in P$ and $v \in Q$ are control parameters varying on sets P and Q compact in R. Regarding the right-hand side of Eq. (1.1) we assume that:

a) $f(t, z, u, v)$ is continuous in $(t, z, u, v) \in X = [0, +\infty) \times R \times P \times Q$;

b) the inequality

$$|f(t, z_1, u, v) - f(t, z_2, u, v)| \leqslant k_* |z_1 - z_2|$$

where k_* is a constant depending only on c, is satisfied for any $u \in P, v \in Q$ and for $t \geqslant 0$, $z_1, z_2 \in R$, $|t| + |z_1| + |z_2| \leqslant c$;

c) a constant B exists such that

$$|(z \cdot f(t, z, u, v))| \leqslant B(1 + |z|^2)$$